Remarks on the Geometry of Wick Rotation in QFT and its Localization on Manifolds

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Abstract

The geometric aspect of Wick rotation in quantum field theory and its localization on manifolds are explored. After the explanation of the notion and its related geometric objects, we study the topology of the set of landing W for Wick rotations and its natural stratification. These structures in two, three, and four dimensions are computed explicitly. We then focus on more details in two dimensions. In particular, we study the embedding of W in the ambient space of Wick rotations, the resolution of the generic metric singularities of a Lorentzian surface Σ by local Wick rotations, and some related S^1-bundles over Σ.

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Contents.

Introduction.

Quantum field theory (QFT) mingles together bold ideas, theoretical concepts, and tricks for practical computations. *Wick rotation* is one such example. In the computation of Feynman diagrams, one constantly faces integrals of meromorphic functions over the Minkowskian space-time or momentum 4-space, which are literally divergent. However by pushing the real time axis into the complex time plane and rotate it to the purely imaginary time axis (Figure I-1), the metric becomes Euclidean and the integrals become ones over Euclidean 4-space. One can then introduce either cutoffs or dimensional regularization to make these integrals finite and next tries to understand the divergent behavior when letting the cutoffs go to infinity, the dimensions of space-time approach 4, or the imaginary time axis rotated back to the real one. Indeed in Wilson's renormalization theory, the attitude of taking the original theory on the Minkowskian space-time as the limit of analytic continuations (via inverse Wick rotation) from the theory on the Euclidean 4-space becomes more profound since the notion of scales and cutoffs is fundamental in his theory and only in Euclidean space do they have better control of (components of) momenta. Due to this origin, the analytic aspect of Wick rotation has been discussed in the literature and the notion has become a standard one in QFT (e.g. [G-J], [I-Z], [P-S]). In contrast, the geometric aspect of it seems dimmed, due to lack of necessity.

However, the study of gravity and string theory forces one to consider manifolds and press on, asking: *How indeed are Lorentzian manifolds Wick-rotated to Riemannian manifolds?* and *How such rotations relate the Minkowskian and the Euclidean theory?* It is for such harder questions that the geometric aspect of Wick rotation should demonstrate its roles.

In this paper, we shall focus on Wick rotation as a geometric procedure that relates metrics of different signatures and study the geometry behind this in its own right. After
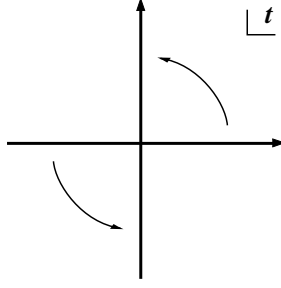
FIGURE I-1. Wick rotation in QFT is a rotation of the real time axis to the imaginary time axis in the complex-time plane.

giving the general definition of Wick rotation for a vector space motivated from above and its local generalization to manifolds in Sec. 1, we study the structure of the set of landing W for Wick rotations in Sec. 2. This set consists of the terminal vector spaces of Wick rotations, on which the metric is required to resume real-valued so that they become the usual real inner product spaces again. The different signatures of the inner product on the terminal spaces that Wick-rotations can lead to induce a stratification of W. After the general study of these in Sec. 2, we compute them explicitly for the cases of two, three, and four demensions in Sec. 3. We then turn our focus to more details of Wick rotations in two dimensions in Sec. 4. We study the embedding of W in the ambient space of Wick rotations, the resolution of the generic metric singularities of a Lorentzian surface Σ by local Wick rotations, and some related S^1-bundles over Σ.

1   Wick rotation and its localization.

In this section we shall formulate a defintion of Wick rotations directly from QFT and discuss its generalizations to manifolds. Readers are referred to [Mi] and [Di2] for some terminology and background from the theory of singularities.

Definition and basic geometric objects involved.

The reflection on Wick rotation in QFT motivates us the following general definition of Wick rotations.

Definition 1.1 [Wick rotation]. Let V be a real vector space with a non-degenerate inner product ρ and $V_{\mathbb{C}} = V \otimes \mathbb{C}$ be its complexification with the natural $\mathbb{C}$-linear inner product $\rho_{\mathbb{C}} = \rho \otimes \mathbb{C}$. Let $\iota : V \hookrightarrow V_{\mathbb{C}}$ be the natural inclusion. Then a *Wick rotation* of V is a smooth family of injective $\mathbb{R}$-linear homomorphisms

$$f_t \ : \ V \ \longrightarrow \ V_{\mathbb{C}} \quad \text{with } t \in [0, 1] \, ,$$

such that $f_0 = \iota$ and that ρ_t defined by $f_t^* \rho$ is non-degenerate on V for all t with ρ_1 real-valued.

Let $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ be the space of all $\mathbb{R}$-linear homomorphisms from V to $V_{\mathbb{C}}$. There are two subsets in $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ that are particularly important in the geometric picture of Wick rotations:

(1) the *set W of landing* that contains all f in $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ with $f^* \rho$ real-valued, and

(2) the *set Ξ of degeneracy* that contains all f in $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ with $f^* \rho$ degenerate.

Both of them are cones in $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ with apex the zero map O. In terms of them, a Wick rotation is simply a path in the complement $Hom_{\mathbb{R}}(V, V_{\mathbb{C}}) - \Xi$ that begins with ι and lands on W.

Notice that $Hom_{\mathbb{R}}(V, V_{\mathbb{C}})$ can be identified with the space $M(n, \mathbb{C})$ of $n \times n$ matrices with complex entries (or of n-tuples of vectors in $\mathbb{C}^n$) once a basis of V is chosen. Also observe that, given n vectors $\xi_1, \ldots, \xi_n$ in $V_{\mathbb{C}}$ with each vector regarded as a column vector, one has the identity

$$\det \left[\rho_{\mathbb{C}}(\xi_i, \xi_j) \right]_{ij} = \left(\det \left[\xi_1, \cdots, \xi_n \right] \right)^2,$$

which follows from the fact that any $n \times n$ matrix is conjugate by an elememt in $SL(n, \mathbb{C})$ to the Jordan form, for which the above identity holds by induction on the rank of elementary Jordan matrices. This implies that the set Ξ is simply the variety in $\mathbb{C}^{n^2}$ described by $\det \left[\xi_1, \cdots, \xi_n \right] = 0$. Thus it is known ([Di1-2]) that Ξ can be naturally stratified by the subset $\Xi_{(r)}$ that consists of $\left[\xi_1, \cdots, \xi_n \right]$ of rank r. Each $\Xi_{(r)}$ is a smooth connected manifold of dimension $n^2 - (n - r)^2$. The singular set of Ξ is the union $\cup_{r=0}^{n-2} \Xi_{(r)}$, which has codimension 3 in Ξ. The complement of Ξ in $M(n, \mathbb{C})$ can now be identified with the general linear group $GL(n, \mathbb{C})$. Since $GL(n, \mathbb{C})$ is path-connected, no matter how two inner products are dressed to V, they can always be connected to each other by a Wick rotation.

Let S^{2n^2-1} be any $(2n^2 - 1)$-sphere in $M(n, \mathbb{C})$ that bounds the zero matrix O_n and is transverse to all the rays from O_n. Since Ξ and W are cones at O_n, to understand them, one may as well study their intersection K_{Ξ} and K_W with S^{2n^2-1}. Since $GL(n, \mathbb{C}) = SL(n, \mathbb{C}) \times \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$, and det is homogeneous, the map

$$\varphi : S^{2n^2-1} - K_{\Xi} \longrightarrow S^1$$

defined by $\varphi = \det / |\det|$ gives a trivial (Milnor) fibration with (Milnor) fiber $SL(n, \mathbb{C})$. And $K_W - K_{\Xi}$ lies in $\varphi^{-1}(\{1, i, -1, -i\})$, a union of four fibers. (Cf. FIGURE 4-4(b).) Consequently, for a physical quantity $\mathcal{F}$ that depends analytically on the inner product, its extension along different Wick rotations from a given inner product to another may assume different values. On the other hand, since

$$\pi_1(SL(n, \mathbb{C})) = \pi_2(SL(n, \mathbb{C})) = 0,$$

a fact that follows either from the property of the Milnor fiber of a normal hypersurface singularity like Ξ ([Di2]) or from some direct argument, besides the set of poles of $\mathcal{F}$ in $GL(n,\mathbb{C})$, the factor S^1, which may be regarded as the generator of $\pi_1(GL(n,\mathbb{C}))$ is the only complication to the analytic extension of $\mathcal{F}$ from the global topology of $GL(n,\mathbb{C})$.

Remark 1.2. In retrospect, one may define a Wick rotation more restrictively as a path that lies in a one-parameter subgroup of $GL(n,\mathbb{C})$ with one end on the identity and the other end on W. This is indeed what happens in QFT.

Generalization to manifolds.

When moving on to a manifold $M = (M, \rho)$, where ρ is a non-degenerate metric on M, the notion of Wick rotations of a vector space indeed can be generalized in at least two ways:

Definition 1.3 [global and local Wick rotation]. (1) *Global* : Let $M_{\mathbb{C}}$ be a complex manifold that admits an embedding $\iota : M \hookrightarrow M_{\mathbb{C}}$ as a totally real submanifold and a non-degenerate $\mathbb{C}$-bilinear metric tensor $\rho_{\mathbb{C}}$ such that, when restricted to $T|_M M_{\mathbb{C}}$, $\rho_{\mathbb{C}}$ is simply the complexification $\rho \otimes \mathbb{C}$. A *global Wick rotation* is then a smooth family of embeddings (or immersions) $f_t : M \to M_{\mathbb{C}}$, $t \in [0,1]$, such that $f_0 = \iota$ and that $f_t^* \rho_{\mathbb{C}}$ is a non-degenerate (complex-valued) inner product tensor on M for all t with $f_1^* \rho_{\mathbb{C}}$ real-valued.

(2) *Local* : QFT teaches us that whenever there is a notion defined on a vector space, one can consider its generalization to a vector bundle by *localizing* the notion. Thus one considers $T_{\mathbb{C}}M$, the complexification $T_* M \otimes \mathbb{C}$ of the tangent bundle, with the inner product tensor $\rho_{\mathbb{C}} = \rho \otimes \mathbb{C}$. Let $\iota : T_* M \to T_{\mathbb{C}}M$ be the natural bundle inclusion. A *local* (or, perhaps better, *fiberwise*) *Wick rotation* is then a smooth family of injective bundle homomorphisms $f_t : T_* M \to T_{\mathbb{C}}M$, $t \in [0,1]$, whose induced maps on M are the identity map, such that $f_0 = \iota$ and $f_t^* \rho_{\mathbb{C}}$ is non-degenerate along each fiber of $T_{\mathbb{C}}M$ for all t with $f_1^* \rho_{\mathbb{C}}$ real-valued. (FIGURE 1-1.)

Remark 1.4. There is a subtlety in the above definition. If M admits a non-degenerate metric, then the type (p,q) of the metric that encodes the dimension of a maximal positive-definite and a maximal negative-definite subspace in each tangent space to M is constant. This can happen if and only if M admits a nowhere degenerate p-plane field and, hence, puts a constraint on the topology of M. In general, one may need to relax the requirement of non-degeneracy of $f_1^* \rho_{\mathbb{C}}$ everythere and allow generic metric singularities to occur, e.g. a discrete set for $f_1^* \rho_{\mathbb{C}}$ Lorentzian. Under this relaxation, $f_1^* \rho_{\mathbb{C}}$ may indeed be a *heterotic metric* on M. The set of metric degeneracy could decompose M into a collection of domains, with the metric restricted to each domain having a different type.

In the following sections, we shall try to understand W better and shall discuss also local Wick rotations for surfaces. Readers note that the identifications $Hom_{\mathbb{R}}(V, V_{\mathbb{C}}) =$
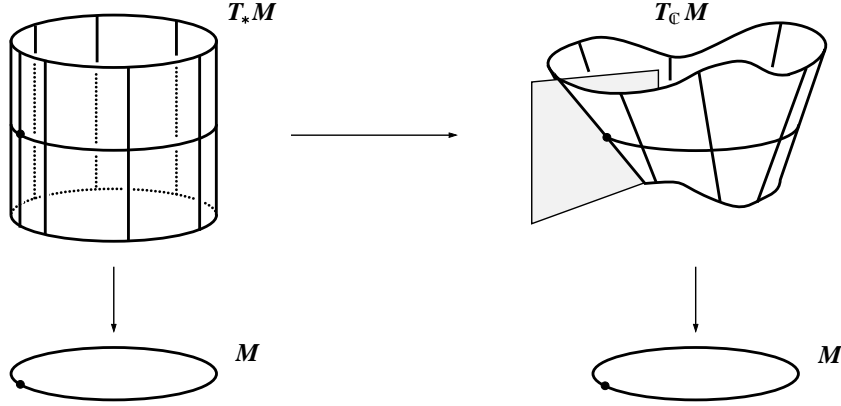
FIGURE 1-1. Wick rotation can be localized on a manifold M. They are fiberwise "rotations" of T_*M in its own complexification $T_{\mathbb{C}}M$, subject to some non-degeneracy requirement. In this picture, only one fiber of $T_{\mathbb{C}}M$ is shown.

$M(n,\mathbb{C})$, $V_{\mathbb{C}} = \mathbb{C}^n = \mathbb{R}^{2n}$, and vectors as column vectors, when a basis is chosen, are implicit in many places; A^t means the transpose of a matrix A. An n-*frame* in this paper will mean an $\mathbb{R}$-*linearly independent* n-tuple of vectors (or vector fields).

2 The set W of landing for Wick rotations.

Since Wick rotations happpn in the complement of Ξ in $M(n,\mathbb{C})$, it is the intersection $W \cap GL(n,\mathbb{C})$ that comes into play. However, it turns out more convenient to study the larger open dense subset in W that consists of all n-frames. For simplicity of notation, we shall denote the latter still by W.

The topology of W.

Let $G_n(\mathbb{R}^{2n})$ be the Grassmann manifold of n-dimensional subspaces in $\mathbb{R}^{2n}$. Recall that there is a tautological $GL(n,\mathbb{R})$-bundle over $G_n(\mathbb{R}^{2n})$. In terms of this, W can be realized as the restriction of this bundle to the subset $\overline{W}$ in $G_n(\mathbb{R}^{2n})$ that consists of all the real n-dimensional subspaces E in $V_{\mathbb{C}}$ with $\rho_{\mathbb{C}}|_E$ real. Our task now is to understand this base space $\overline{W}$.

Let $\mathbb{C}^* = \mathbb{C} - \{0\}$. Recall the inclusion $\iota : V \hookrightarrow V_{\mathbb{C}}$ and the $\mathbb{C}^*$-action on $V_{\mathbb{C}}$ by multiplication. Let $V = V_- \oplus V_+$ be an orthogonal decomposition of V with $\rho|_{V_-}$ negative-definite and $\rho|_{V_+}$ positive-definite. Let $(e_1, \ldots, e_n)$ be a corresponding orthonormal basis for V with, say, the first p of them in V_- and the rest in V_+. Let

$$V_1 = e^{-i\frac{3\pi}{4}} \cdot V_- + e^{-i\frac{\pi}{4}} \cdot V_+ = Span_{\mathbb{R}}(u_1, \ldots, u_n)$$

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and
$$V_2 \;=\; i \cdot V_1 \;=\; Span_{\mathbb{R}}(u_1', \ldots, u_n'),$$

where $u_j = e^{-i\frac{3\pi}{4}} e_j$ (resp. $e^{-i\frac{\pi}{4}} e_j$) if $e_j \in V_-$ (resp. V_+) and $u_j' = i\, u_j$. Observe that $\rho_{\mathbb{C}}|_{V_1}$ is purely imaginary negative-definite while $\rho_{\mathbb{C}}|_{V_2}$ is purely imaginary positive-definite.

Let $v = (v_1, v_2)$ with respect to the decomposition $V_{\mathbb{C}} = V_1 \oplus V_2$. Then, with respect to the basis $(u_1, \ldots, u_n; u_1', \ldots, u_n')$,

$$\rho_{\mathbb{C}}(v, v) \;=\; 2v_1^t v_2 \,+\, i\,(v_2^t v_2 - v_1^t v_1).$$

Consequently, if one defines $Q(v) = \rho_{\mathbb{C}}(v, v)$ and lets $\overline{\rho}$ be the positive-definite inner product on $V_{\mathbb{C}}$ that takes $(u_1, \ldots, u_n; u_1', \ldots, u_n')$ as an orthonormal frame, then

$$Q^{-1}(\mathbb{R}) \;=\; \{\, (v_1, v_2) \in V_{\mathbb{C}} \,|\, \overline{\rho}(v_1, v_1) = \overline{\rho}(v_2, v_2) \,\} \,.$$

A real n-dimensional subspace E in $V_{\mathbb{C}}$ has the restriction $\rho_{\mathbb{C}}|_E$ real-valued if and only if E is contained in $Q^{-1}(\mathbb{R})$. Such E is characterized by the following lemma.

Lemma 2.1. $E \subset Q^{-1}(\mathbb{R})$ *if and only if E is realizable as the graph of an isometry from* $(V_1, \overline{\rho})$ *to* $(V_2, \overline{\rho})$.

Proof. From the expression of $\rho_{\mathbb{C}}(v, v)$ in terms of (v_1, v_2), the if-part is clear. For the only-if part, observe that if $v = (v_1, v_2)$ is in $Q^{-1}(\mathbb{R})$, then $\overline{\rho}(v_1, v_1) = \overline{\rho}(v_2, v_2)$. Consequently, if v in $Q^{-1}(\mathbb{R})$ is non-zero, then both v_1 and v_2 must be non-zero. This implies that both the projections $\mathrm{pr}_1 : E \to V_1$ and $\mathrm{pr}_2 : E \to V_2$ induced from those of $V_{\mathbb{C}}$ into V_1 and V_2 are isomorphisms. The map $\mathrm{pr}_2 \circ \mathrm{pr}_1^{-1}$ from $(V_1, \overline{\rho})$ to $(V_2, \overline{\rho})$ is then an isometry whose graph is E. This completes the proof.

$\square$

Since $\overline{\rho}$ is positive-definite, one has

Corollary 2.2. *The set W of landing for Wick rotations is a natural $GL(n, \mathbb{R})$-bundle over $O(n)$, the real orthogonal group in dimension n.*

We shall say more about the quadratic function Q in Sec. 4. For now let us study some details of $\overline{W}$.

A natural stratification of $\overline{W}$.

The space $\overline{W}$ can be decomposed into a collection of strata. Each stratum contains all E in $\overline{W}$ whose inner product $\rho_{\mathbb{C}}$ is of a fixed type. With respect to the basis $(u_1, \ldots, u_n)$ for V_1 and $(u_1', \ldots, u_n')$ for V_2, the isometry from V_1 to V_2 whose graph gives E can be represented by an element $A \in O(n)$. Furthermore if letting $x \in \mathbb{R}^n$ be the coordinates on E with respect to the basis $(\mathrm{pr}_1^{-1} u_j)_j$ (recall the proof of Lemma 2.1), one has

$$Q(v) \;=\; 2\, x^t A x \;=\; x^t \,(A + A^t)\, x \,,$$

for v in E. In this way, the type (r,s) of $\rho_{\mathbb{C}}|_E$ corresponds exactly to the number of positive and negative eigenvalues of the symmetrization $(A+A^t)/2$ of A. From this, one can show that

Proposition 2.3. *Let* $\overline{W}_+ = SO(n)$ *and* $\overline{W}_- = O_-(n)$ *be the two components of* $\overline{W}$. *Let* $\overline{W}_{r,s}$ *be the subset in* $\overline{W}$ *that contains all* E, *on which* $\rho_{\mathbb{C}}|_E$ *is of type* (r,s). *Then* $\overline{W}_{r,s}$ *is path-connected except* $\overline{W}_{0,0}$, *which has two components. Together they form a stratification of* $\overline{W}$ *with the following stratum relations, where* $A \leftarrow B$ *means that* B *is contained in the closure of* A.

$n:\ odd$

$$SO(n):\quad \overline{W}_{n,0} \leftarrow \overline{W}_{n-2,0} \leftarrow \overline{W}_{n-4,0} \leftarrow \ \cdots\ \leftarrow \overline{W}_{1,0}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-2,2} \leftarrow \overline{W}_{n-4,2} \leftarrow \ \cdots\ \leftarrow \overline{W}_{1,2}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-4,4} \leftarrow \ \cdots\ \leftarrow \overline{W}_{1,4}$$
$$\cdots\qquad\qquad\cdots$$
$$\downarrow\qquad\quad \downarrow$$
$$\overline{W}_{3,n-3} \leftarrow \overline{W}_{1,n-3}$$
$$\downarrow$$
$$\overline{W}_{1,n-1}\ ,$$

$$O_-(n):\quad \overline{W}_{n-1,1} \leftarrow \overline{W}_{n-3,1} \leftarrow \overline{W}_{n-5,1} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,1}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-3,3} \leftarrow \overline{W}_{n-5,3} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,3}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-5,5} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,5}$$
$$\cdots\qquad\qquad\cdots$$
$$\downarrow\qquad\quad \downarrow$$
$$\overline{W}_{2,n-2} \leftarrow \overline{W}_{0,n-1}$$
$$\downarrow$$
$$\overline{W}_{0,n}\ .$$

$n:\ even$

$$SO(n):\quad \overline{W}_{n,0} \leftarrow \overline{W}_{n-2,0} \leftarrow \overline{W}_{n-4,0} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,0}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-2,2} \leftarrow \overline{W}_{n-4,2} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,2}$$
$$\downarrow\qquad\quad \downarrow\qquad\quad \downarrow$$
$$\overline{W}_{n-4,4} \leftarrow \ \cdots\ \leftarrow \overline{W}_{0,4}$$
$$\cdots\qquad\qquad\cdots$$
$$\downarrow\qquad\quad \downarrow$$

$$\overline{W}_{2,n-2} \leftarrow \overline{W}_{0,n-2}$$
$$\downarrow$$
$$\overline{W}_{0,n} \ ,$$

$$O_-(n): \qquad \overline{W}_{n-1,1} \leftarrow \overline{W}_{n-3,1} \leftarrow \overline{W}_{n-5,1} \leftarrow \quad \cdots \quad \leftarrow \overline{W}_{1,1}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\overline{W}_{n-3,3} \leftarrow \overline{W}_{n-5,3} \leftarrow \quad \cdots \quad \leftarrow \overline{W}_{1,3}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\overline{W}_{n-5,5} \leftarrow \quad \cdots \quad \leftarrow \overline{W}_{1,5}$$
$$\cdots \qquad\qquad \cdots$$
$$\downarrow \qquad\qquad \downarrow$$
$$\overline{W}_{3,n-3} \leftarrow \overline{W}_{1,n-3}$$
$$\downarrow$$
$$\overline{W}_{1,n-1} \ .$$

Remark 2.4. Notice that the multiplication by i in $V_{\mathbb{C}}$ induces a homeomorphism between $\overline{W}_{r,s}$ and $\overline{W}_{s,r}$.

Proof. Let us outline the idea of the proof first. The $SO(n)$-action on $O(n)$ by conjugation commutes with symmetrization; thus it leaves each $\overline{W}_{r,s}$ invariant. Since $SO(n)$ is path-connected, each orbit of this action, i.e. a conjugacy class in $O(n)$, also has to be path-connected. Consequently, the quotient map for this action, $O(n) \to O(n)/\sim$ pushes the decomposition $\{\overline{W}_{r,s}\}$ of $\overline{W}$ down to the decomposition $\{\overline{W}(r,s)/\sim\}$ of $\overline{W}/\sim$ with the same stratum relations; and the corresponding pieces $\overline{W}_{r,s}$ and $\overline{W}_{r,s}/\sim$ have the same number of components. This reduces the problem to the study of $O(n)/\sim$ with the quotient decomposition $\{\overline{W}(r,s)/\sim\}$. To accomplish this, a key fact is that [Hu]

$$SO(n)/\sim \ = \ T/\Gamma,$$

where T is a maximal torus in $SO(n)$ and Γ is the Weyl group acting on T. Up to conjugations, elements in T is built up from elements in $U(1)$, represented by the 2×2 blocks $D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, whose symmetrization is $\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}$. This gives a stratification of $U(1)$ by the sign of $\cos\theta$. The stratification of $\overline{W}/\sim$ is induced from the product of these stratified $U(1)$'s; hence so is that of $\overline{W}$. This also shows that the jump of the type of $\rho_{\mathbb{C}}|_E$ for E in various strata of $\overline{W}$ is a multiple of 2. (FIGURE 2-1.)

Let us now look at the details.

Case (i): n odd. For $n = 2k+1$, we may choose T consisting of the block diagonal matrices:

$$T \ = \ \{Diag(1, D(\theta_1), \cdots, D(\theta_k))|\theta_i \in [0, 2\pi]\}$$
$$\cong \ \underbrace{S^1 \times \cdots \times S^1}_{k};$$

$(0,0)$
$\theta = {}^{\pi}\!/_2$

$(-,-)$ $(+,+)$
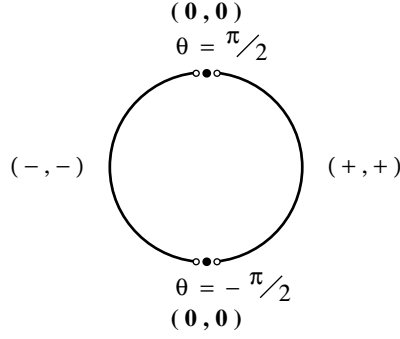
$\theta = -{}^{\pi}\!/_2$
$(0,0)$

FIGURE 2-1. The fundamental stratification of $U(1)$ whose prod-
uct leads to the stratification of $\overline{W}$. The influence to the type of
metrics is also indicated.

then Γ consists of the $2^k k!$ permutations of the indices of $(\theta_1, \cdots, \theta_k)$ composed with
substitutions

$$(\theta_1, \cdots, \theta_k) \ \longrightarrow \ (\epsilon_1 \theta_1, \cdots, \epsilon_k \theta_k),$$

where $\epsilon_i = \pm 1$. It follows that the quotient $SO(2k+1)/\!\sim$ is stratified by the collection of
subsets $I_{\alpha\beta}$ defined by

$$\begin{aligned} I_{\alpha\beta} \ = \ & \{(\theta_1, \cdots, \theta_k) | 0 \leq \theta_1 \leq \cdots \leq \theta_\alpha \\ & < \frac{\pi}{2} = \theta_{\alpha+1} = \cdots = \theta_{k-\beta} < \theta_{k-\beta+1} \leq \cdots \leq \theta_k \leq \pi \}. \end{aligned}$$

One can check that $I_{\alpha\beta}$ is path-connected and lies in $\overline{W}_{2\alpha+1,2\beta}/\!\sim$. From the symmetry
$(r,s) \leftrightarrow (s,r)$, one also obtains the stratum structure in question for $O_-(2k+1)$. By
comparing the even- or oddness of r and s, one can see that $\overline{W}(r,s)/\!\sim$ lies either in
$SO(n)$ or $O_-(n)$; thus it must be that

$$I_{\alpha\beta} \ = \ \overline{W}_{2\alpha+1,2\beta}/\!\sim.$$

This shows that $\overline{W}_{2\alpha+1,2\beta}/\!\sim$ is path-connected. One easily sees that the relations among
these strata, which are the same as those among $I_{\alpha\beta}$, are as indicated in the Proposition.
This completes the proof for n odd.

Case (ii): n even. For $n = 2k$, consider first the $SO(2k)$-part, we may choose

$$T \ = \ \{Diag\,(D(\theta_1), \cdots, D(\theta_k)) | \theta_i \in [0, 2\pi]\},$$

then Γ consists of the $2^{k-1} k!$ permutations of the indices of $(\theta_1, \cdots, \theta_k)$ composed with
substitutions

$$(\theta_1, \cdots, \theta_k) \ \longrightarrow \ (\epsilon_1 \theta_1, \cdots, \epsilon_k \theta_k)$$

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with $\epsilon_i = \pm 1$ and $\epsilon_1 \cdots \epsilon_k = 1$. From these data, it follows that

$$
\begin{aligned}
T/\Gamma \;&\cong\; \Delta_1 \cup_h \Delta_2, \quad \text{where} \\
\Delta_1 \;&=\; \{(\theta_1, \cdots, \theta_k) | 0 \le \theta_1 \le \cdots \le \theta_k \le \pi\}, \\
\Delta_2 \;&=\; \{(\theta_1, \cdots, \theta_k) | \pi \le \theta_k \le 2\pi,\ 0 \le \theta_1 \le \cdots \le \theta_{k-1} \le 2\pi - \theta_k\}
\end{aligned}
$$

and h is the pasting map from $\{\theta_1 = 0\}$-face $\cup\ \{\theta_k = \pi\}$-face of Δ_1 to that of Δ_2 defined by

$$
\begin{aligned}
h(0, \theta_2, \cdots, \theta_{k-1}, \theta_k) \;&=\; (0, \theta_2, \cdots, \theta_{k-1}, 2\pi - \theta_k) \quad \text{and} \\
h(\theta_1, \cdots, \theta_{k-1}, \pi) \;&=\; (\theta_1, \cdots, \theta_{k-1}, \pi).
\end{aligned}
$$

The two collections of subsets: $I_{\alpha\beta}$ in Δ_1 and $I'_{\alpha\beta}$ in Δ_2 defined by

$$
\begin{aligned}
I_{\alpha\beta} \;=\; \{(\theta_1, \cdots, \theta_k) | 0 \le \theta_1 \le \cdots \le \theta_\alpha \\
< \frac{\pi}{2} = \theta_{\alpha+1} = \cdots = \theta_{k-\beta} < \theta_{k-\beta+1} \le \cdots \le \theta_k \le \pi\}
\end{aligned}
$$

and

$$
I'_{\alpha\beta} \;=\; \text{the reflection of } I_{\alpha\beta} \text{ with respect to the hyperplane } \theta_k = \pi\,,
$$

form a stratum structure for T/Γ after being pasted along the map h. By the same reason as in the case of n odd, $I_{\alpha\beta}$ and $I'_{\alpha\beta}$ together constitute $\overline{W}_{2\alpha, 2\beta}/\sim$. Thus $\overline{W}_{2\alpha, 2\beta}/\sim$ is disconnected if and only if both $I_{\alpha\beta}$ and $I'_{\alpha\beta}$ have empty intersection with $\{\theta_1 = 0\}$ and $\{\theta_k = \pi\}$. This implies that $(\frac{\pi}{2} \le \theta_1 \le \pi)$ and $(0 \le \theta_k \le \frac{\pi}{2}$ or $\frac{3\pi}{2} \le \theta_k \le 2\pi)$. Together with the defining inequalities for Δ_1 and Δ_2, we have either $\theta_1 = \theta_2 = \cdots = \theta_k = \frac{\pi}{2}$ or $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = \frac{\pi}{2}$, $\theta_k = \frac{3\pi}{2}$, which correspond to I_{00} and I'_{00} respectively. Thus $\overline{W}_{r,s}/\sim$ is path-connected except $\overline{W}_{0,0}/\sim$, which has two components.

For the $O_-(2k)$ part, any A in $O_-(2k)$ is conjugate under $SO(2k)$ to some $Diag(-1, B)$, where $B \in SO(2k-1)$. Thus $O_-(2k)/\sim$ is actually a quotient of T'/Γ', where T' is a maximal torus of $SO(2k-1)$ and Γ' is the Weyl group acting on T'. The stratum structure descends then from that for n odd. As already shown in the odd case, each stratum in T'/Γ' is path-connected; hence so is its quotient. This implies that all the corresponding $\overline{W}_{r,s}/\sim$ in $O_-(2k)$ are path-connected. The stratum relations also follow. This completes the case of n even and hence the proof of the proposition.

$\square$

Remark 2.5. For a manifold $M = (M, \rho)$, one can apply the above discussion fiberwise and construct the *bundle of landing* $W(M)$ and the $O(n)$-bundle $\overline{W}(M)$ over M. The latter has the structure group $O(n)$, which acts on the $O(n)$-fiber by conjugation. Consequently, $\overline{W}(M)$ has two components $\overline{W}_+(M)$, whose fiber is $SO(n)$, and $\overline{W}_-(M)$, whose fiber is $O_-(n)$. The stratification of $\overline{W}$ is invariant under conjugation; and hence it leads a stratification of $\overline{W}(M)$ by subbundles $\overline{W}_{r,s}(M)$ whose fiber is $\overline{W}_{r,s}$.

So far our attention has been mainly to the set of landing W itself. However a complete geometric picture for Wick rotations involves also how W embeds in $M(n, \mathbb{C})$ or $GL(n, \mathbb{C})$.

This should be a problem that falls well in the realm of the study of singularities of hypersurface intersections and their complement in $\mathbb{C}^n$. In Sec. 4, we shall study this problem in dimension two.

3 $\overline{W}$ in two, three, and four dimensions.

As corollaries and supplements to the previous section, let us provide concrete calculations of $\overline{W}$ at two, three, and four dimensions. We shall denote $\overline{W}$ for V of dimension n by $\overline{W}(n)$ and its two components by $\overline{W}_+(n)$ - the $SO(n)$-part - and $\overline{W}_-(n)$ - the $O_-(n)$-part.

In two dimensions.

It is clear that the topology and stratification of $\overline{W}(2)$ is given by

$$\overline{W}_+(2): \qquad \overline{W}_{2,0} \cong \overset{\circ}{I} \quad \leftarrow \quad \overline{W}_{0,0} \cong \partial I$$
$$\downarrow$$
$$\overline{W}_{0,2} \cong \overset{\circ}{I}$$

$$\overline{W}_-(2): \qquad \overline{W}_{1,1} \cong S^1 \qquad\qquad\qquad ,$$

where $\overset{\circ}{I}$ is an open interval (FIGURE 3-1).

(0 , 0)

(− , −)   **(+ , +)**
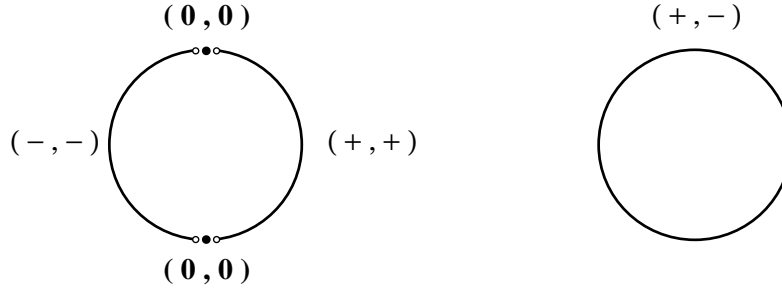
(0 , 0)

(+ , −)

FIGURE 3-1.   The stratification of $\overline{W}(2)$ and the corresponding types of inner products.

In three dimensions.

Recall that $SO(3) = \mathbb{R}\mathrm{P}^3$ has a model as the closed 3-ball D^3 of radius π with the antipodal points on the boundary ∂D^3 identified. With respect to the coordinates $(\mathbf{n}, \theta)$,

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where $\theta \in [0, \pi]$ and $\mathbf{n} \in S^2$, $(\mathbf{n}, \theta)$ represents the rotation along the axis $\mathbf{n}$ by an angle θ following the right-hand rule. It is conjugate to $Diag(1, D(\theta))$. For $O_-(3)$, the same model still works with small modifications: the $\mathbf{n}$ is now the (-1)-eigenvector of an element in $O_-(3)$ and $\theta = \pi$, which corresponds to $-$Identity, is now at the origin of the ball. Thus $(\mathbf{n}, \theta)$ is conjugate to $Diag(-1, D(\theta))$. From these, $\overline{W}_+(3)$ with its stratification is given by

$$\overline{W}_+(3): \quad \overline{W}_{3,0} \cong \overset{\circ}{D}_{\frac{\pi}{2}} \quad \leftarrow \quad \overline{W}_{1,0} \cong S^2$$
$$\downarrow$$
$$\overline{W}_{1,2} \cong \mathbb{R}\mathrm{P}^3 - D_{\frac{\pi}{2}} \quad ,$$

where $D_{\frac{\pi}{2}}$ is the closed 3-ball of radius $\frac{\pi}{2}$ in our model and $\overset{\circ}{D}_{\frac{\pi}{2}}$ is its interior. And the $\overline{W}_-(3)$ part can be obtained by the symmetry $(r, s) \leftrightarrow (s, r)$. (FIGURE 3-2.)
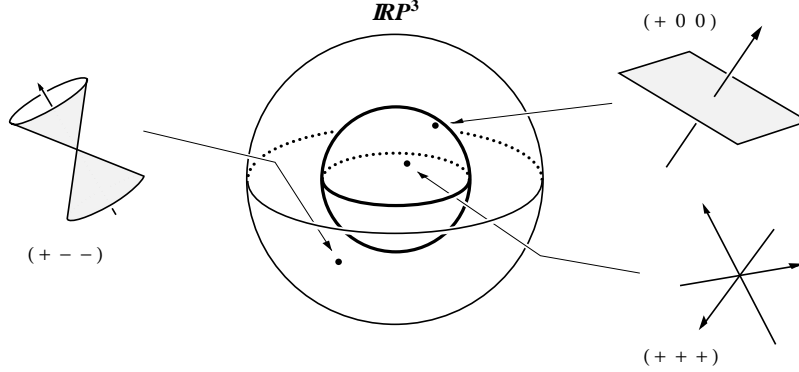
FIGURE 3-2. The stratification of $\overline{W}_+(3)$ and the various corresponding types of inner product. In the picture, $\overline{W}_+(3) = \mathbb{R}\mathrm{R}^3$ is realized as a 3-ball with each pair of antipodal points on the boundary identified. The $\overline{W}_-(3)$-part is similar with $\pm$ changed to $\mp$.

In four dimensions.

(a) The $\overline{W}_+(4)$ part. Recall that $SO(4) \cong S^3 \times SO(3)$ can be interpreted as the group of $\mathbb{R}$-linear, orientation and norm-preserving self-maps on quaternions. In the above topological product decomposition, the S^3-factor represents the unit quaternion group, which acts on quaternions by left multiplication; the $SO(3)$-factor represents the isotropy group of 1, which fixes the real axis and rotates the $\mathbb{R}^3$-space spanned by i, j, and k. In the following discussion, we shall not distinguish the normed space of quaternions and the Euclidean 4-space $\mathbb{E}^4$ with $(1, i, j, k)$ as an orthonormal basis.

Consider the coset $S^3 \cdot h$ in $SO(4)$ with $h \in SO(3)$. Represent h by $(\mathbf{n}, \theta)$ as introduced in the discussion for $\overline{W}(3)$. Given $q \in S^3$, let

(1) $a = \operatorname{Re} q$, the real part of q;

(2) $b = \langle q, \mathbf{n} \rangle$, where both q and $\mathbf{n}$ are regarded as vectors in $\mathbb{E}^4$; and

(3) A be the matrix in $SO(4)$ that represents qh.

Notice that $a^2 + b^2 \le 1$. It follows from straightforward computation that $\frac{A + A^t}{2}$ is conjugate to the diagonal matrix $Diag\,(\cos\theta_1, \cos\theta_1, \cos\theta_2, \cos\theta_2)$ with

$$\cos\theta_1 \;=\; \frac{1}{2}\left(a + a\cos\theta - b\sin\theta + \sqrt{(a - a\cos\theta + b\sin\theta)^2 + 4(1 - a^2 - b^2)\sin^2\frac{\theta}{2}}\right)$$

and

$$\cos\theta_2 \;=\; \frac{1}{2}\left(a + a\cos\theta - b\sin\theta - \sqrt{(a - a\cos\theta + b\sin\theta)^2 + 4(1 - a^2 - b^2)\sin^2\frac{\theta}{2}}\right).$$

Using this, consider the following maps:

$$S^3 \;\overset{\eta_{(\mathbf{n},\theta)}}{\longrightarrow}\; D^2 \;\overset{\zeta}{\longrightarrow}\; \Delta$$
$$q \;\longmapsto\; (a,b) \;\longmapsto\; (\cos\theta_1, \cos\theta_2) \quad,$$

where D^2 is the unit disk and $\Delta = \{(s,t)|s,t \in [-1,1], s \ge t\}$. For $\theta \in (0,\pi)$, one has in D^2

$$\{\cos\theta_1 = 0\} \;=\; \{a\cos\frac{\theta}{2} - b\sin\frac{\theta}{2} = -\sin\frac{\theta}{2}, \quad a^2 + b^2 \le 1\},$$
$$\{\cos\theta_2 = 0\} \;=\; \{a\cos\frac{\theta}{2} - b\sin\frac{\theta}{2} = \sin\frac{\theta}{2}, \quad a^2 + b^2 \le 1\}.$$

For $\theta = 0$, one has

$$\{\cos\theta_1 = 0\} \;=\; \{\cos\theta_2 = 0\} \;=\; \{a = 0,\ b \le 1\}.$$

For $\theta = \pi$, one has

$$\{\cos\theta_1 = 0\} \;=\; \{\cos\theta_2 = 0\} \;=\; \{(0,-1),(0,1)\}.$$

Decomposing D^2 by $(sign\,(\cos\theta_1), sign\,(\cos\theta_2))$ under the map ζ and considering the preimage under $\eta_{(\mathbf{n},\theta)}$ of the components of the decomposition, one then has:

(i) $\overline{W}_{4,0} \cong \overline{W}_{0,4} \cong$ a $\overset{\circ}{D}{}^3$ bundle over $\overset{\circ}{D}{}^3$, which must be $\overset{\circ}{D}{}^3 \times \overset{\circ}{D}{}^3 \cong \overset{\circ}{D}{}^6$. The base manifold $\overset{\circ}{D}{}^3$ is $\mathbb{RP}^3 - \{\theta = \pi\}$.

(ii) $\overline{W}_{2,2} \cong$ a two-punctured S^3 bundle over a one-punctured $\mathbb{RP}^3$. For $\theta \ne 0, \pi$, the two punctures on each fiber correspond to two sections in the trivial S^3 bundle, $(\mathbb{RP}^3 - \{\theta = 0, \pi\}) \times S^3$: one comes from

$$\sigma_1 \;:\; \mathbb{RP}^3 - \{\theta = 0, \pi\} \;\longrightarrow\; S^3$$
$$(\mathbf{n}, \theta) \;\longmapsto\; \eta^{-1}_{(\mathbf{n},\theta)}(\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2})$$

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and the other comes from

$$\sigma_2 \;:\; \mathbb{R}\mathrm{P}^3 - \{\theta = 0, \pi\} \;\longrightarrow\; S^3$$
$$(\mathbf{n}, \theta) \;\longmapsto\; \eta_{(\mathbf{n},\theta)}^{-1}(-\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2})\,.$$

Since $(\mathbf{n}, \pi) = (-\mathbf{n}, \pi)$,

$$\lim_{\theta \to \pi} \sigma_1(\mathbf{n}, \theta) \;=\; -\lim_{\theta \to \pi} \sigma_2(\mathbf{n}, \theta)\,.$$

Thus σ_1, σ_2 together with their extension over $\{\theta = \pi\}$ form a connected double covering X over $\mathbb{R}\mathrm{P}^3 - \{\theta = 0\}$. Thus $X \cong S^2 \times (-1, 1)$ and $\overline{W}_{2,2} \cong$ the complement bundle of X in $(\mathbb{R}\mathrm{P}^3 - \{\theta = 0\}) \times S^3$.

(iii) $\overline{W}_{2,0} \cong \overline{W}_{0,2} \cong$ the bundle over $\mathbb{R}\mathrm{P}^3 - \{\theta = 0, \pi\}$ with fibre the boundary of the fibre of the trivial bundle $\overline{W}_{4,0}$; hence it is homeomorphic to $(\overset{\circ}{D^3} - \{\theta = 0\}) \times S^2$.

(iv) $\overline{W}_{0,0}$ lies over $\{\theta = 0\} \cup \{\theta = \pi\}$. Over $\theta = 0$, it is the preimage of $\{a = 0, |b| \le 1\}$ under $\eta_{(\mathbf{n}_0, 0)}$, where $\mathbf{n}_0$ is any fixed direction. This is a 2-sphere. Over $\theta = \pi$, it is a connected double covering over $\mathbb{R}\mathrm{P}^2$. Thus it is also a 2-sphere. Consequently, $\overline{W}_{0,0}$ is the disjoint union $S^2 \amalg S^2$.

(b) The $\overline{W}_-(4)$ part. Let $A = qh$ be a matrix in $O_-(4)$ that lies in the coset $S^3 \cdot h$, where $h \in O_-(3)$. With the similar notation and argument as in the $\overline{W}_+(4)$ part ($\mathbf{n}$ is now the (-1)-eigenvector of h), $\frac{A + A^t}{2}$ is conjugate to $Diag\,(-1, 1, \cos\xi, \cos\xi)$, where $\cos\xi = a\cos\theta - b\sin\theta$. Similar to the $\overline{W}_+(4)$ part, one has maps

$$S^3 \;\overset{\eta_{(\mathbf{n},\theta)}}{\longrightarrow}\; D^2 \;\overset{\zeta}{\longrightarrow}\; [-1, 1]$$
$$q \;\longmapsto\; (a, b) \;\longmapsto\; \cos\xi \quad.$$

and also the decomposition of D^2 by $sign\,(\cos\xi)$. The preimage under $\eta_{(\mathbf{n},\theta)}$ of the $(+)$-region (resp. the $(-)$-region) is the hemisphere with center $\eta_{(\mathbf{n},\theta)}^{-1}(\cos\theta, -\sin\theta)$, (resp. $\eta_{(\mathbf{n},\theta)}^{-1}(-\cos\theta, \sin\theta)$).

Let

$$\sigma \;:\; \mathbb{R}\mathrm{P}^3 \;\longrightarrow\; S^3$$
$$(\mathbf{n}, \theta) \;\longmapsto\; \eta_{(\mathbf{n},\theta)}^{-1}(\cos\theta, -\sin\theta) \quad.$$

It is straightforward to check that σ is well-defined and is of degree 1. Regard $\overline{W}_{3,1}$ as a subbundle in the trivial bundle $\mathbb{R}\mathrm{P}^3 \times S^3$ over $\mathbb{R}\mathrm{P}^3$ with coordinates (x, y), $x \in \mathbb{R}\mathrm{P}^3$, $y \in S^3$. Recall the unit quaternion group structure on S^3. The left multiplication then takes a hemisphere to another hemisphere. Let Ω now be the open hemisphere in S^3 centered at 1. The map

$$\overline{W}_{3,1} \;\longrightarrow\; \mathbb{R}\mathrm{P}^3 \times \Omega$$
$$(x, y) \;\longmapsto\; (x, \sigma(x)^{-1} \cdot y)$$

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gives a bundle homeomorphism which trivializes $\overline{W}_{3,1}$. Thus

$$\overline{W}_{3,1} \;\cong\; \overline{W}_{1,3} \;\cong\; \mathbb{R}\mathrm{P}^3 \times \overset{\circ}{D}{}^3\ .$$

Followingly, one also has

$$\overline{W}_{1,1} \;\cong\; \mathbb{R}\mathrm{P}^3 \times S^2\ .$$

This completes the discussion for $\overline{W}(4)$.

4   Wick rotations in two dimensions.

In this last section, we shall study more the geometric objects associated to Wick rotations in two dimensions. Following the notations from Sec. 1 and 2, we recall from Sec. 2 and 3 that every (ξ_1, ξ_2) in $W(2)$ lies in a real 2-subspace E in $V_{\mathbb{C}}$ that is realizable as the graph of an isometry ϕ from $(V_1, \overline{\rho})$ to $(V_2, \overline{\rho})$. In two dimensions, two different such E_1 and E_2 can have non-trivial intersection only if one is in $\overline{W}_+(2) = SO(2)$ and the other in $\overline{W}_-(2) = O_-(2)$. The two points in $\overline{W}_+(2)$ labelled by $(0,0)$ in FIGURE 3-1 correspond to the two E's in $\overline{W}$ that happen to be a complex line. From Corollary 2.2, $W(2)$ with all the $\mathbb{R}$-linearly dependent (ξ_1, ξ_2) removed is a trivial $GL(2, \mathbb{R})$-bundle over $\overline{W}(2)$. Let $W_+(2)$ be the part over $\overline{W}_+(2)$ and $W_-(2)$ be the part over $\overline{W}_-(2)$. Then, from Sec. 3, we know that $W(2) - \Xi$ has six components, four for $W_+(2) - \Xi$ and two for $W_-(2) - \Xi$. However, all together, $W(2)$ is a connected subset in $M(2, \mathbb{C})$. Indeed, since any non-zero $v \in Q^{-1}(\mathbb{R})$ determines exactly one orientation-preserving and one orientation-reversing isometry from $(V_1, \overline{\rho})$ to $(V_2, \overline{\rho})$, the intersection of $W_+(2)$ and $W_-(2)$ is exactly the set of linearly dependent (ξ_1, ξ_2)'s in $W(2)$. (Cf. FIGURE 4-4.)

How $W(2)$ embeds in $M(2, \mathbb{C})$.

Recall from Sec. 1 the Milnor fibration φ from $S^7 - K_\Xi$ to S^1 for dimension two. The determinant function det is now a nondegenerate quadratic polynomial in four variables and hence Ξ has only an isolated A_1-singularity at the origin O. The Milnor fiber $SL(2, \mathbb{C})$ is homeomorphic to the tangent bundle $T_* S^3 = S^3 \times \mathbb{R}^3$. Its boundary in S^7 is K_Ξ, which is homeomorphic to the unit tangent bundle $T_1 S^3 = S^3 \times S^2$. The union $\varphi^{-1}(e^{i\theta}) \cup K_\Xi \cup \varphi^{-1}(-e^{i\theta})$ for every θ is a smooth $S^3 \times S^3$ embedded in S^7. More explicitly, if one lets Δ be the diagonal of $S^3 \times S^3$ parametrized by (p,p), $p \in S^{n-1}$, and $\overline{\Delta}$ be the anti-diagonal parametrized by $(p, \overline{p})$, where $\overline{p}$ is the antipodal point of p, then $\varphi^{-1}(e^{i\theta})$ forms the $\frac{\pi}{2}$-neighborhood of Δ while $\varphi^{-1}(-e^{i\theta})$ forms the $\frac{\pi}{2}$-neighborhood of $\overline{\Delta}$, where the radius $\frac{\pi}{2}$ is measured at a (p,p) or $(p, \overline{p})$, either horizontally or vertically in $S^3 \times S^3$, from the latitude in S^3 that takes p-$\overline{p}$ as north-south poles FIGURE 4-1. All these either are or follow from well-known ([Di2]) properties of A_1-singularities and S^3. Thus, from Sec. 1, K_W lies in $\varphi^{-1}(\{\pm 1, \pm i\}) \cup K_\Xi$, which is the union of two $S^3 \times S^3$ pasted along K_Ξ (FIGURE 4-1,
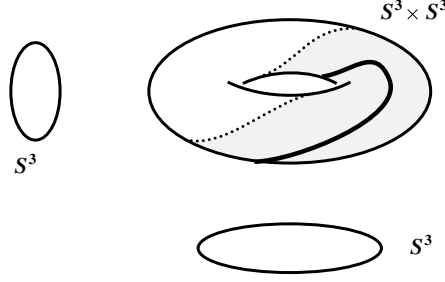
FIGURE 4-1. How $\varphi^{-1}(e^{i\theta})$, $\varphi^{-1}(-e^{i\theta})$, and K_Ξ, for a given θ occupy the product $S^3 \times S^3$ is illustrated. The set $\varphi^{-1}(e^{i\theta})$ is indicated by the shaded part, a $\frac{\pi}{2}$-neighborhood of the diagonal.

4-4(b)). The $S^3 \times S^3$ associated $\varphi^{-1}(\{\pm 1\})$ contains the base of the cone $W_+(2)$ while the $S^3 \times S^3$, associated to $\varphi^{-1}(\{\pm i\})$, contains the base of the cone $W_-(2)$.

In this particular dimension, there is another way to see how $W(2)$ embeds in $M(2,\mathbb{C})$, using the fact ([Ro]) that

$$S^{p+q+1} = S^p * S^q,$$

where we recall that the *join* $X * Y$ of two topological spaces X and Y is defined to be the space $X \times Y \times [0,1]$ with the following identifications:

    (i) $(x, y, 0) \sim (x, y', 0)$ for all $x \in X$, $y, y' \in Y$; and

    (ii) $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$, $y \in Y$.

The two generating spheres, S^p and S^q, of the join is embedded in the resulting S^{p+q+1} with linking number 1 and this gives a fibration of S^{p+q+1} over $[0,1]$ with generic fiber $S^p \times S^q$, which collapses to S^p over $\{0\}$ and to S^q over $\{1\}$. (FIGURE 4-2). Let us now employ this to our problem.

Up to a slight modification, we assume that ρ is positive-definite. Hence $V = \mathbb{R}^2$ with the standard orthonormal basis (e_1, e_2) and coordinates (x_1, x_2). We shall adapt ourselves to the following identifications: $V_\mathbb{C} = \mathbb{C}^2 = \mathbb{R}^4$ with coordinates

$$\xi = (z_1, z_2)^t = (x_1 + i\,y_1,\ x_2 + i\,y_2)^t;$$

and $M(2,\mathbb{C}) = \mathbb{C}^4 = \mathbb{R}^8$ with coordinates

$$\begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} = \begin{pmatrix} x_1 + i\,y_1 & u_1 + i\,v_1 \\ x_2 + i\,y_2 & u_2 + i\,v_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & u_1 & v_1 \\ x_2 & y_2 & u_2 & v_2 \end{pmatrix}.$$

We also regard $M(2,\mathbb{C})$ as $V_\mathbb{C} \oplus V_\mathbb{C}$ with metric $\overline{\rho} \oplus \overline{\rho}$. We shall identify the S^7 with the unit sphere at the origin in any of the above identifications.
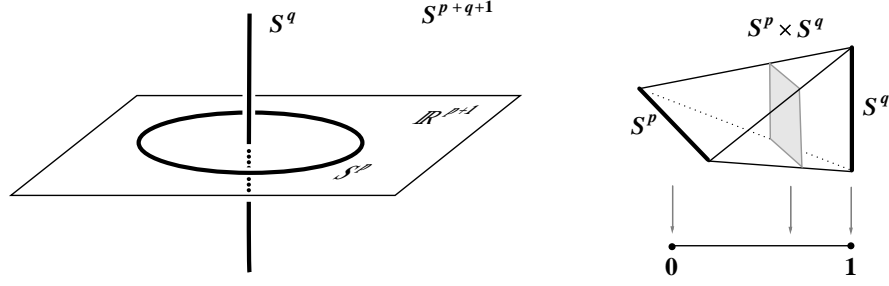
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FIGURE 4-2. The realization of S^{p+q+1} as the join $S^p * S^q$ (cf. [Ro], p. 5) and the associated fibration of S^{p+q+1} over $[0,1]$ (cf. FIGURE 4-3). In the picture on the left, S^{p+q+1} is regarded as $\mathbb{R}^{p+q+1} \cup \{\infty\}$.

In terms of these coordinates, K_W is described by the following system:

$$x_1 y_1 + x_2 y_2 = 0, \quad u_1 v_1 + u_2 v_2 = 0$$

$$x_1 v_1 + y_1 u_1 + x_2 v_2 + y_2 u_2 = 0$$

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 + u_1^2 + v_1^2 + u_2^2 + v_2^2 = 1 \,.$$

Perform now the following change of coordinates, which is an isometry with respect to $\overline{\rho} \oplus \overline{\rho}$,

$$
\begin{aligned}
x_1 &= \frac{\overline{x}_1 - \overline{x}_2 + \overline{u}_1 + \overline{u}_2}{2}, & u_1 &= \frac{\overline{x}_1 + \overline{x}_2 - \overline{u}_1 + \overline{u}_2}{2}, \\
y_1 &= \frac{-\overline{x}_1 - \overline{x}_2 - \overline{u}_1 + \overline{u}_2}{2}, & v_1 &= \frac{-\overline{x}_1 + \overline{x}_2 + \overline{u}_1 + \overline{u}_2}{2}, \\
x_2 &= \frac{\overline{y}_1 - \overline{y}_2 + \overline{v}_1 + \overline{v}_2}{2}, & u_2 &= \frac{\overline{y}_1 + \overline{y}_2 - \overline{v}_1 + \overline{v}_2}{2}, \\
y_2 &= \frac{-\overline{y}_1 - \overline{y}_2 - \overline{v}_1 + \overline{v}_2}{2}; & v_2 &= \frac{-\overline{y}_1 + \overline{y}_2 + \overline{v}_1 + \overline{v}_2}{2},
\end{aligned}
$$

the system then becomes: (with the $^-$ removed for simplicity of notations in all the rest of discussions)

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = \tfrac{1}{2}, \qquad u_1^2 + v_1^2 + u_2^2 + v_2^2 = \tfrac{1}{2},$$

$$x_1 u_1 + y_1 v_1 + x_2 u_2 + y_2 v_2 = 0,$$

$$x_1^2 + y_1^2 = u_2^2 + v_2^2, \qquad x_2^2 + y_2^2 = u_1^2 + v_1^2.$$

Let $\widehat{S}_r^3$ be the 3-sphere $x_1^2 + y_1^2 + x_2^2 + y_2^2 = r^2$ in $\mathbb{R}^4 \times \{0\}$ and $\overline{S}_r^3$ be the 3-sphere $u_1^2 + v_1^2 + u_2^2 + v_2^2 = r^2$ in $\{0\} \times \mathbb{R}^4$; then $S^7 = \widehat{S}_1^3 * \overline{S}_1^3$ with K_W lying in the generic leaf $\widehat{S}_{\frac{1}{\sqrt{2}}}^3 \times \overline{S}_{\frac{1}{\sqrt{2}}}^3$. The expression of the system suggests also a fibration of K_W over, say, $\widehat{S}_{\frac{1}{\sqrt{2}}}^3$. To illuminate this, let us denote $\widehat{S}_{\frac{1}{\sqrt{2}}}^3$ (resp. $\overline{S}_{\frac{1}{\sqrt{2}}}^3$) simply by $\widehat{S}^3$ (reps. $\overline{S}^3$) and introduce

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the coordinates $(R; r, \theta_1, \theta_2; \overline{r}, \overline{\theta}_1, \overline{\theta}_2)$ for S^7 with

$$
\begin{aligned}
x_1 &= r \cos\theta_1\,, & u_1 &= \overline{r} \cos\overline{\theta}_1\,, \\
y_1 &= r \sin\theta_1\,, & v_1 &= \overline{r} \sin\overline{\theta}_1\,, \\
x_2 &= \sqrt{R^2 - r^2} \cos\theta_2\,, & u_2 &= \sqrt{1 - R^2 - \overline{r}^2} \cos\overline{\theta}_2\,, \\
y_2 &= \sqrt{R^2 - r^2} \sin\theta_2\,, & v_2 &= \sqrt{1 - R^2 - \overline{r}^2} \sin\overline{\theta}_2\,,
\end{aligned}
$$

where $R \in [0,1]$, $r \in [0, R]$, $\overline{r} \in [0, \sqrt{1 - R^2}]$, and $\theta_1, \theta_2, \overline{\theta}_1, \overline{\theta}_2 \in [0, 2\pi)$. Then

$$
\widehat{S}^3 \times \overline{S}^3 \;=\; \{\, R = \tfrac{1}{\sqrt{2}} \,\}\,.
$$

In $\widehat{S}^3$, the set $\{r = 0\} \cup \{r = \frac{1}{\sqrt{2}}\}$ describes a link while all other $\{r = \text{constant} \in (0, \frac{1}{\sqrt{2}})\}$ describes a torus $\widehat{T}_r$; and similarly for the torus $\overline{T}_{\overline{r}}$ defined in $\overline{S}^3$. These come simply from the realization of S^3 as the join $S^1 * S^1$. (FIGURE 4-3.)
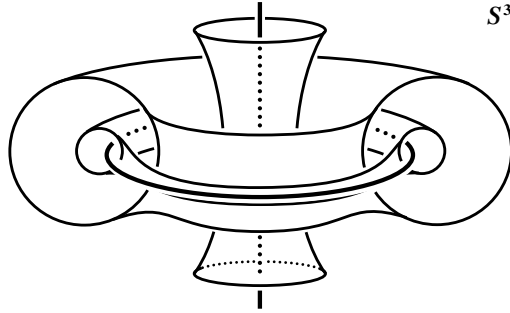

S^3

FIGURE 4-3. The foliation of S^3 by tori with two degenerate leaves that form a link (cf. [P-R], p. 62). In this picture, S^3 is realized as $\mathbb{R}^3 \cup \{\infty\}$ and part of the tori shown is excised for clarity.

The system that describes K_W now becomes

$$
R = \tfrac{1}{\sqrt{2}}\,, \quad r^2 + \overline{r}^2 = \tfrac{1}{2}\,,
$$
$$
r\,\overline{r}\,\cos\!\big(\tfrac{(\overline{\theta}_2 - \theta_2) + (\overline{\theta}_1 - \theta_1)}{2}\big)\cos\!\big(\tfrac{(\overline{\theta}_2 - \theta_2) - (\overline{\theta}_1 - \theta_1)}{2}\big) \;=\; 0\,,
$$

whose solution is given by

$$
R = \tfrac{1}{\sqrt{2}}\,,
$$
$$
\text{and} \qquad (r, \overline{r}) \;=\; (0, \tfrac{1}{\sqrt{2}})\,, \quad \text{or} \quad (\tfrac{1}{\sqrt{2}}, 0)\,,
$$
$$
\text{or} \qquad (\overline{\theta}_2 - \theta_2) \;\equiv\; (\overline{\theta}_1 - \theta_1) + \pi \quad (\text{mod } 2\pi)
$$
$$
\text{or} \qquad (\overline{\theta}_2 - \theta_2) \;\equiv\; -(\overline{\theta}_1 - \theta_1) + \pi \quad (\text{mod } 2\pi)\,.
$$

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Given (r, θ_1, θ_2) in $\widehat{S}^3$ with $r \in (0, \frac{1}{\sqrt{2}})$, the third equation describes the $(1,1)$-curve C_-, and the fouth equation the $(1,-1)$-curve C_+, through (θ_1, θ_2) in the torus $\overline{T}_{\overline{r}}$, where $\overline{r} = \sqrt{\frac{1}{2} - r^2}$ (FIGURE 4-4). Notice that C_+ and C_- intersects also at $(\theta_1 + \pi, \theta_2 + \pi)$ in $\overline{T}_{\overline{r}}$. For $r = 0$ or $\frac{1}{\sqrt{2}}$ in $\widehat{S}^3$, the solution is respectively the circle $\overline{r} = \frac{1}{\sqrt{2}}$ or $\overline{r} = 0$ in $\overline{S}^3$. Together, this gives a fibration of K_W over $\widehat{S}^3$ with generic fiber two circles meeting at two points, which degenerates into a single circle over the link $\{r = 0, r = \frac{1}{\sqrt{2}}\}$ in $\overline{S}^3$.

On the other hand, in terms of $(R; r, \theta_1, \theta_2; \overline{r}, \overline{\theta}_1, \overline{\theta}_2)$, the determinant function det on $M(2, \mathbb{C})$ restricted to S^7 is given by

$$
\det|_{S^7} = \left[r\sqrt{R^2 - r^2} \sin(\theta_2 - \theta_1) + \overline{r}\sqrt{1 - R^2 - \overline{r}^2} \sin(\overline{\theta}_2 - \overline{\theta}_1) \right]
$$
$$
+ i \left[r\overline{r} \sin(\overline{\theta}_1 - \theta_1) - \sqrt{R^2 - r^2}\sqrt{1 - R^2 - \overline{r}^2} \sin(\overline{\theta}_2 - \theta_2) \right],
$$

whose restriction to K_W can be written as

$$
\det|_{K_W} = r\overline{r} \sin(\tfrac{\overline{\theta}_2 - \overline{\theta}_1 + \theta_2 - \theta_1}{2}) \cos(\tfrac{(\overline{\theta}_2 - \theta_2) - (\overline{\theta}_1 - \theta_1)}{2})
$$
$$
+ i r\overline{r} \sin(\tfrac{(\overline{\theta}_1 - \theta_1) - (\overline{\theta}_2 - \theta_2)}{2}) \cos(\tfrac{(\overline{\theta}_2 - \theta_2) + (\overline{\theta}_1 - \theta_1)}{2}).
$$

When restricted to $\overline{T}_{\overline{r}}$, observe that the zero-set of the real part of $\det|_{K_W}$ consists of C_- and the $(1,1)$-curve

$$
(\overline{\theta}_2 - \theta_2) \equiv (\overline{\theta}_1 - \theta_1) - 2(\theta_2 - \theta_1) \pmod{2\pi}
$$

while the zero-set of the imaginary part consists of C_+ and the $(1,1)$-curve

$$
(\overline{\theta}_2 - \theta_2) \equiv (\overline{\theta}_1 - \theta_1) \pmod{2\pi}.
$$

This shows that C_+ lies in $W_+(2)$ while C_- lies in $W_-(2)$ except for their intersection, whicht lies in Ξ. Unless $\theta_2 - \theta_1 \equiv \pm\frac{\pi}{2} \pmod{2\pi}$, these loci divide $\overline{T}_{\overline{r}}$ into six regions, as indicated in FIGURE 4-4(a). It follows also immediately that the degenerate part of the fibration of K_W over $\widehat{S}^3$, which are the two tori in K_W described respectively by $r = 0$ and $r = \frac{1}{\sqrt{2}}$, are in K_Ξ.

From these data, one observes that, in addition to $C_- \cap C_+$, the locus $\det|_{K_W} = 0$ intersects $C_- \cup C_+$ also at the following two points

$$
(\overline{\theta}_1 - \theta_1, \overline{\theta}_2 - \theta_2) = (-(\theta_1 - \theta_2) + \tfrac{\pi}{2}, (\theta_1 - \theta_2) + \tfrac{\pi}{2})
$$
$$
\text{or} \quad (-(\theta_1 - \theta_2) + \tfrac{3\pi}{2}, (\theta_1 - \theta_2) + \tfrac{3\pi}{2}) \pmod{2\pi}
$$

in C_+. They differ from $C_- \cap C_+$ unless $\theta_1 - \theta_2 \equiv \pm\frac{\pi}{2} \pmod{2\pi}$. In the generic situation when these two points do not coincide with $C_- \cap C_+$, they correspond to (ξ_1, ξ_2) in K_W that happens to span a complex line. Following from the indication in FIGURE 4-4(a), for a generic (r, θ_1, θ_2) with $r \in (0, \frac{1}{\sqrt{2}})$, these four special points decompose $C_+ \cup C_-$ into six arcs. Each lies in a different component of $K_W - K_\Xi$, exhausting the six components of
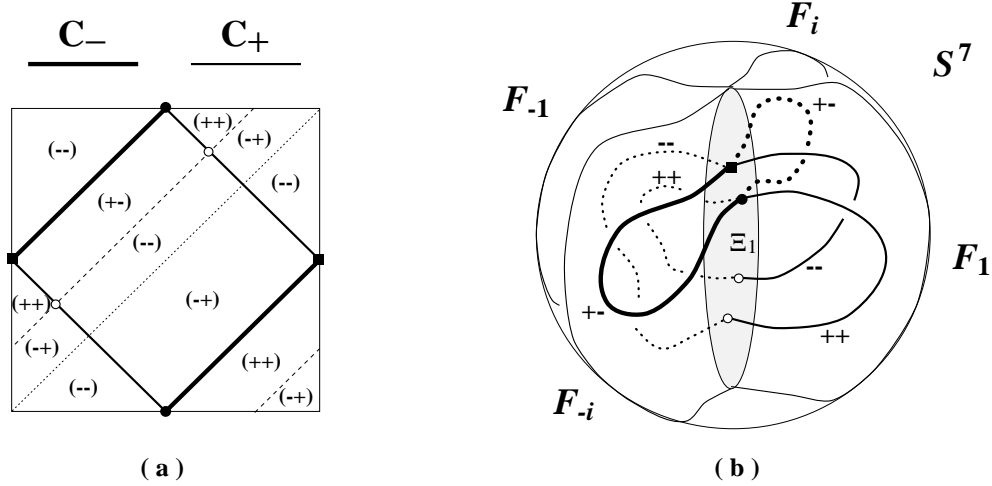
FIGURE 4-4. In (a), the torus $\overline{T}_{\overline{r}}$ is realized as a rectangle with the two pairs of parallel edges identified. Its coordinates are given in terms $\overline{\theta}_1 - \theta_1$ (horizontal axis) and $\overline{\theta}_2 - \theta_2$ (vertical axis) that ranges from 0 to 2π, either rightwards or upwards. The locus Re det $|_{K_W} = 0$ consists of - - - - - - and C_-. The locus Im det $|_{K_W} = 0$ consists of $\cdots\cdots$ and C_-. In (b), the four Milnor fibers $\varphi^{-1}(1)$, $\varphi^{-1}(i)$, $\varphi^{-1}(-1)$, $\varphi^{-1}(-i)$ in S^7 that contain K_W are indicated and denoted by F_1, F_i, F_{-1}, F_{-i} respectively. Their boundary meet at K_Ξ (denoted by Ξ_1 in the picture). The way the generic fiber $C_- \cup C_+$ of K_W over $\widehat{S}^3$ travels around these four Milnor fibers is shown. The type of the corresponding metrics for each piece is indicated, as follows from (a). (Cf. FIGURE 3-1.)

$K_W - K_{\Xi}$. From the $(\pm, \pm)$ associated to each reagion, where the first $\pm$ indicates the sign of the real part of det and the second the sign of the imaginary part of det, one obtains also how they travel around in the Milnor fibers, as indicated in FIGURE 4-4(b).

This concludes our discussion for the embedding of $W(2)$ in $M(2, \mathbb{C})$.

Local Wick rotations of a surface.

Having said much about Wick rotations, let us now consider local Wick rotations of a surface Σ. The following proposition shows that any generic Lorentzian metric on Σ can be inverse-Wick-rotated to a Riemannian metric. Hence its metric singularities are resolved by (inverse) local Wick rotations.

Proposition 4.1 [transitivity]. *Let $\Sigma = (\Sigma, \rho_0)$ be a compact Riemannian surface. Let ρ_1 be a generic Lorentzian metric on Σ. Then there is a local Wick rotation $f_t : T_*\Sigma \to T_{\mathbb{C}}\Sigma$, $t \in [0, 1]$, of Σ such that $f_1^* \rho_{0\mathbb{C}} = \rho_1$.*

Proof. Let $\varsigma = \{p_1, \ldots, p_s\}$ be the set of singularities of ρ_1, where $s = |\chi(\Sigma)|$ is the absolute value of the Euler characteristic of Σ, and Δ_i be a collection of disjoint disks around p_i. From linear algebra, there exist a pair of line fields (X_+, X_-) on $\Sigma - \varsigma$ that are simultaneously orthogonal with respect to ρ_0 and ρ_1 such that ρ_1 is positive-definite on X_+ and negative-definite on X_-. Observe that for any z_+, z_- non-zero in $\mathbb{C}$, $z_+ X_+$ and $z_- X_-$ remain orthogonal in $(T_{\mathbb{C}}(\Sigma - \varsigma), \rho_{0\mathbb{C}})$. Consequently, if one lets $c_+ = \rho_1 / \rho_0$ along X_+, $c_- = -\rho_1 / \rho_0$ along X_-, and v_+, v_- be locally-defined nowhere-vanishing vector fields that lie respectively in X_+ and X_-, then the family f_t from $T_*\Sigma$ to $T_{\mathbb{C}}\Sigma$ defined by, for example,

$$ f_t \; : \; (v_+, v_-) \; \longrightarrow \; \left((1 - t + t\sqrt{c_+})\, v_+ \, , \; (1 - t + t\sqrt{c_-})\, e^{\frac{\pi}{2} it}\, v_- \right) $$

is singular over ς but is globally well-defined on $\Sigma - \varsigma$ and is a local Wick rotation from ρ_0 to ρ_1 on $\Sigma - \varsigma$. To make f_t, $t \neq 1$, an injective bundle homomorphism from the whole $T_*\Sigma$ into $T_{\mathbb{C}}\Sigma$, one can consider a smooth family of bump fuctions μ_i^t, $t \in [0, 1]$, that take values in $[0, 1]$ with $\mu_i^t(p_i) = 1$ and support in $\Delta_i \cap B(p_i, 1 - t)$, where $B(p_i, 1 - t)$ is the ball at p_i of radius $1 - t$ with respect to ρ_0. Let $\mu^t = \sum_i \mu_i^t$. One can then redefine f_t by $\mu^t \iota + (1 - \mu^t) f_t$, where recall that ι is the natural inclusion from $T_*\Sigma$ to $T_{\mathbb{C}}\Sigma$. In the limit, we have a bundle homomorphism f_1 from $T_*\Sigma$ to $T_{\mathbb{C}}\Sigma$ that is injective over $\Sigma - \varsigma$ and is the zero-map at ς such that $f_1^* \rho_{0\mathbb{C}} = \rho_1$. This completes the proof.

$\square$

Remark 4.2. It is conceivable that, with a careful study of the set of singularities of generic metrics on a manifold M - it is the set of singularities of some p-plane field on M and has codimension at least one -, the above assertion and proof should work also at general dimensions. Also notice that for M an orientable 3-manifold, T_*M is trivial and M admits non-singular metrics of types $(+, -, -)$ and $(-, +, +)$. Any such metric can be obtained from a local Wick rotation of a Riemannian one.

Remark 4.3. Any metrics on Σ can be thought of as representatives of their conformal classes. Hence the above construction gives also a local Wick rotation from a given Riemann surface to a given Lorentz surface of the same topology. In [Li] we discussed some details of Lorentz surfaces and speculated the possible formulation of Lorentzian conformal field theory (CFT), following Atiyah and Segal's definition. Understanding how the usual (Riemannian) CFT is transformed under local Wick rotations should give hints to how Lorentzian CFT can be constructed. This demands works in the future.

Among the several bundles over Σ that appear in the geometry of local Wick rotations, let us take a look at the $\overline{W}(2)$-bundle, which concerns the landing of local Wick rotations.

Recall ([Sc], [St]) that an S^1-bundle Y over a compact surface Σ is classified by its first Stiefel-Whitney class $w_1(Y)$ in $H^1(\Sigma; \mathbb{Z}_2)$, if $\partial\Sigma$ is non-empty, and by w_1 and its Euler class $e(Y)$ in $H^2(\Sigma; \{\mathbb{Z}\})$ if Σ is closed, where $\{\mathbb{Z}\}$ is the local coefficient $\mathbb{Z}$ on Σ that is plainly $\mathbb{Z}$ for Σ orientable and is twisted by the local orientation of Σ for Σ non-orientable. With respect to a triangulation of Σ and up to a bundle isomorphism that descends to the identity map on Σ, w_1 determines the S^1-bundle Y over the 1-skeleton. For Σ with non-empty boundary this is enough to determine the whole bundle. For Σ closed, $e(Y)$ tells in addition how the part of Y over the 2-skeleton is pasted to the part over the 1-skeleton. In terms of these, we have the following proposition that classifies $\overline{W}(\Sigma)$.

Proposition 4.4. *Given a Riemannian surface $\Sigma = (\Sigma, \rho)$. The total space of both components of $\overline{W}(\Sigma)$, i.e. $\overline{W}_+(\Sigma)$ and $\overline{W}_-(\Sigma)$, are orientable. As S^1-bundles over Σ, their structures are as listed in the following table:*

		w_1	e
$\partial\Sigma \neq \emptyset$:	$\overline{W}_+(\Sigma)$	$w_1(\Sigma)$	—
	$\overline{W}_-(\Sigma)$	$w_1(\Sigma)$	—
$\partial\Sigma = \emptyset$:	$\overline{W}_+(\Sigma)$	$w_1(\Sigma)$	0
	$\overline{W}_-(\Sigma)$	$w_1(\Sigma)$	$2\,e(\Sigma)$,

where $w_1(\Sigma) = w_1(T_*\Sigma)$ (vanishes if Σ is orientable) and $e(\Sigma) = e(T_*\Sigma)$ are respectively the Stiefel-Whitney and the Euler class of Σ.

Proof. First it is straightforward to check that the orientability of the fibre of each bundle along any loop coincides with the orientability of Σ along the same loop. This gives the w_1-column. It implies also that the total space of $\overline{W}(\Sigma)$ are orientable.

For Σ closed orientable, $\overline{W}_+(\Sigma) \cong \Sigma \times S^1$; thus $e(\overline{W}_+(\Sigma)) = 0$. As for $\overline{W}_-(\Sigma)$, since $w_1(\overline{W}_-(\Sigma)) = w_1(\Sigma) = 0$ from the previous discussion, one can fix a triangulation of Σ and a trivialization of $\overline{W}_-(\Sigma)$ over the 1-skeleton. Let $T_1\Sigma$ be the unit tangent bundle of Σ. As $SO(2)$-bundles, the transition function for $T_1\Sigma$ is by left multiplication while that

for $\overline{W}_-(\Sigma)$ is by conjugation. The following comparison of the two

$$\left(\begin{array}{cc} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{array}\right)\left(\begin{array}{cc} \cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & -\cos\theta_0 \end{array}\right)\left(\begin{array}{cc} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{array}\right)$$
$$= \left(\begin{array}{cc} \cos(2\alpha+\theta_0) & \sin(2\alpha+\theta_0) \\ \sin(2\alpha+\theta_0) & -\cos(2\alpha+\theta_0) \end{array}\right),$$

while

$$\left(\begin{array}{cc} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{array}\right)\left(\begin{array}{cc} \cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & -\cos\theta_0 \end{array}\right) = \left(\begin{array}{cc} \cos(\alpha+\theta_0) & \sin(\alpha+\theta_0) \\ \sin(\alpha+\theta_0) & -\cos(\alpha+\theta_0) \end{array}\right),$$

tells us that the pasting homomorphism of $W_-(\Sigma)$ over the 2-skeleton to the part over the 1-skeleton winds twice as many as that for $T_1\Sigma$. Thus $e(\overline{W}_-(\Sigma)) = 2e(\Sigma)$.

For Σ closed non-orientable, consider its orientation cover $\widetilde{\Sigma}$ with pullback metric $\widetilde{\rho}$. Then $\overline{W}_\pm(\widetilde{\Sigma})$ is the pullback bundle of $\overline{W}_\pm(\Sigma)$ via the covering map. By the functorial property of the Euler class and the result for Σ orientable, we have also that $e(\overline{W}_-(\Sigma)) = 2e(\Sigma)$. This completes the proof.

$\square$

Remark 4.5. For an orientable 3-manifold M^3, since T_*M^3 is trivial, all the $W(3)$-, $\overline{W}$-, etc. bundles over M^3, that arise from local Wick rotations, are trivial.

Having discussed some of the geometric aspects of Wick rotations, one certainly likes to see its feedback to QFT, gravity, and string theory, from which the notion arises. We shall leave that for future works.

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